Nowhere zero flows in graphs

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Nowhere zero Γ-flows

Let $\vec{G}$ be a directed graph and Γ be an abelian group. Then a **nowhere zero Γ-flow** for $\vec{G}$ is an assignment of the nonzero elements of Γ to the edges of $\vec{G}$ such that the net flow entering each vertex is zero.

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Lemma (Tutte, 1950)

Let G be a graph and Γ be an abelian group. Then $\vec{G}$ has a nowhere zero Γ-flow for some orientation $\vec{G}$ of G if and only if $\vec{G}$ has a nowhere zero Λ-flow for all orientations $\vec{G}$ of G and all abelian groups Λ with $|\Lambda| \geq |\Gamma|$.

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We say G has a nowhere zero Γ-flow if some (or equivalently, if every) orientation of G has a nowhere zero Γ-flow.

Nowhere zero k-flows

Let G be a graph and $k \geq 2$ be an integer. Then a **nowhere zero k-flow** for G is a nowhere zero $\mathbb{Z}$-flow (for some fixed orientation of G) in which each edge is assigned one of the integers $\pm 1, \pm 2, \ldots, \pm(k-1)$.

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Lemma (Tutte, 1950)

A graph G has a nowhere zero k-flow if and only if it has a nowhere zero $\mathbb{Z}_k$-flow.

Duality

Lemma (Tutte, 1950)

Let G be a plane graph, G^* be its planar dual and Γ be an abelian group with $|\Gamma| = k$. Then the following statements are equivalent.

- G has a nowhere zero Γ-flow.
- G is k-face-colourable.
- G^* is k-vertex-colourable.

Nowhere zero 2- and 3-flows

Lemma

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It is NP-complete to decide if a graph has a nowhere zero 3-flow. (A planar graph has a nowhere zero 3-flow if and only if its planar dual is 3-vertex colourable.)

Conjecture (Tutte 1972)

Every 4-edge-connected graph has a nowhere zero 3-flow.

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Theorem (Jaeger 1979)

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Theorem (Seymour 1981)

Every bridgeless graph has a nowhere zero 6-flow.

Γ-connected graphs

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Theorem (Thomassen, Wu, Zhang 2012+)

Every 6-edge-connected graph is $\mathbb{Z}_3$-connected (and hence has a nowhere zero 3-flow).

Flow Polynomials

Given a graph G and an abelian group Γ let $F_G(\Gamma)$ be the number of distinct nowhere zero Γ-flows for some fixed orientation of G.

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Suppose e is an edge of G.
If e is a loop then $F_G(\Gamma) = (|\Gamma| - 1)F_{G-e}(\Gamma)$.
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We can similarly deduce that $F_G(q)$ is a polynomial in q.

WARNING: the number of nowhere zero q-flows is in general NOT equal to $F_G(q)$. So this notation is misleading.

Flow roots

Tutte's 5-flow conjecture is equivalent to saying that $F_G(q) > 0$ for all integers $q \geq 5$ and all bridgeless graphs G. Since $F_G(q)$ is a polynomial in q we can also consider its real or complex zeros.

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Theorem (BJ 2003)

If G is bridgeless with n vertices then $F_G(q) > 0$ for all real $q > 2 \log_2 n$.